# LECTURE 3

## THE MAIN RESULTS

## FOR

## $C^m(\mathbb{R}^n)\big|_E$

## ($E$ FINITE)

# NOTATION

Fix $m, n \geqslant 1$.

$$X = C^m(\mathbb{R}^n)$$

$$\|F\|_X = \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|$$

---

$E \subset \mathbb{R}^n \Rightarrow$

$$X(E) = \{ F|_E : F \in X \}$$

$$\|f\|_{X(E)} = \inf \{ \|F\|_X : F \in X,\ F = f \text{ on } E \}$$

$$J_x(F) = \left[ (m-1)^{\underline{rst}} \text{ DEGREE TAYLOR POLY OF } F \text{ AT } x \right]$$

$$J_x(F)(y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y-x)^\alpha$$

$$J_x(F) \in \mathcal{P}$$

$\mathcal{P}$ = VECTOR SPACE OF ALL $(m-1)^{\underline{rst}}$ DEGREE (REAL) POLYS ON $\mathbb{R}^n$.

RECALL, $X = C^m(\mathbb{R}^n)$.

$$\boxed{\text{CONSTANTS DENOTED } c, C, C', \ldots \text{ ALWAYS DEPEND ONLY ON } m, n.}$$

- $A \lesssim B$ MEANS $A \leq CB$
- $A \sim B$ MEANS $cA \leq B \leq CA$.
- (A & B HAVE THE "SAME ORDER OF MAG.")

# THEOREMS FROM LECTURE 1

## (AND A FEW MORE)

LET $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

**THM 1**: Given $f: E \to \mathbb{R}$,

we can compute the order of magnitude of $\|f\|_{X(E)}$

using $\leq CN \log N$ computer ops.

and at most $CN$ memory.

("COMPUTATION OF THE NORM")

THM 2: Given $f : E \to \mathbb{R}$,

WE CAN EFFICIENTLY* COMPUTE A FUNCTION $F \in X$ SUCH THAT

$F = f$ on $E$

AND

$$\|F\|_X \leq C \|f\|_{X(E)}$$

$$* \left( \begin{array}{ll} \text{ONE-TIME WORK} \leq & CN \log N \\ \text{QUERY WORK} \leq & C \log N \\ \text{STORAGE} \leq & CN \end{array} \right)$$

"COMPUTE EXTENSION"

Given $f : E \to \mathbb{R}$, $x \in E$, $M > 0$,

WE DEFINED THE (POSSIBLY EMPTY)

CONVEX SET

$$\boxed{\begin{aligned}&\Gamma(x, M) = \\ &\quad \left\{ J_x(F) : F \in X, \|F\|_X \leq M, \ F = f \text{ on } E \right\}.\end{aligned}}$$

WE WANT TO COMPUTE THE APPROXIMATE SIZE AND SHAPE OF $\Gamma(x, M) \subset \mathcal{P}$.

TO DO SO, WE WILL DEFINE

CONVEX SETS $\Gamma_\ell(x, M) \subset \mathcal{P}$

FOR $x \in E$, $M > 0$, $\ell = 0, 1, 2, \cdots$

BY INDUCTION ON $\ell$.

The $\Gamma_\ell(x, M)$ WILL HAVE THE

FOLLOWING KEY PROPERTIES:

$$\boxed{\Gamma_\ell(x, M) \supset \Gamma(x, M)}$$

$$\boxed{\Gamma_\ell(x, M) \supset \Gamma_{\ell+1}(x, M)}$$

FOR LARGE ENOUGH $\ell_*$ (DEPENDING ONLY ON $m, n$),

$\Gamma_{\ell_*}(x, M)$ HAS APPROXIMATELY THE SAME SIZE & SHAPE AS $\Gamma(x, M)$.

# THE INDUCTION DEFINING

# $\Gamma_\ell(x, M)$

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BASE CASE : $\ell = 0$

For $x \in E$, DEFINE

$$\Gamma_0(x, M) = \{ P \in \mathcal{P} : |\partial^\alpha P(x)| \leq M \text{ (all } |\alpha| \leq m-1) \; \& \; P(x) = f(x) \}$$

Then $\Gamma_0(x, M)$ is a (possibly empty) convex set, and

$\Gamma_0(x, M) \supset \Gamma(x, M)$.

## Induction Step:

Fix $\ell \geq 0$ and $M > 0$.

Suppose we have already defined

$\Gamma_\ell(x, M) \subset \mathcal{P}$ for all $x \in E$.

We will define

$\Gamma_{\ell+1}(x, M) \subset \mathcal{P}$ for all $x \in E$.

DEFINITION :

$\Gamma_{\ell+1}(x, M)$ CONSISTS OF ALL

$P \in \Gamma_\ell(x, M)$ WITH THE

FOLLOWING PROPERTY:

Given $y \in E$ there exists $P' \in \Gamma_\ell(y, M)$ such that

$$|\partial^\alpha (P - P')(x)| \le M |x-y|^{m-|\alpha|}$$

for $|\alpha| \le m-1$

So, as promised,

we have defined $\Gamma_\ell(x, M)$.

An easy induction on $\ell$

using Taylor's Thm shows

that

$$\boxed{\Gamma_\ell(x, M) \supset \Gamma(x, M).}$$

Also, the definition of $\Gamma_{\ell+1}$

shows at once that

$$\boxed{\Gamma_\ell(x, M) \supset \Gamma_{\ell+1}(x, M).}$$

<u>THM 3</u>: FOR A LARGE ENOUGH $\ell_*$ (DEPENDING ONLY ON $m, n$)

WE HAVE

$$\Gamma_{\ell_*}(x, M) \supset \Gamma(x, M) \supset \Gamma_{\ell_*}(x, CM).$$

"COMPUTATION OF THE $\Gamma$'s."

<u>COROLLARY</u>: If $\Gamma_{\ell_*}(x, M) \neq \phi$ for some $x \in E$, then $\|f\|_{X(E)} \leq CM$.

THM 4: Let $E \subset \mathbb{R}^n$, $\#(E) < \infty$.

Then there exists a linear map

$T : X(E) \to X$

such that for any $f \in X(E)$

we have

$$Tf = f \text{ on } E$$

and

$$\| Tf \|_X \leq C \| f \|_{X(E)} .$$

WE CAN TAKE $T$ TO HAVE

A SIMPLE FORM

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Given $x \in \mathbb{R}^n$, we can write

$$Tf(x) = \sum_{y \in E} \lambda(x,y) f(y)$$

for coefficients $\lambda(x,y)$ indep. of $f$.

For each $x$, we will have at most

$C$ nonzero $\lambda(x,y)$

("BOUNDED DEPTH")

T CAN BE COMPUTED EFFICIENTLY.

AFTER ONE-TIME WORK $\leq C N \log N$

USING STORAGE $\leq C N$,

WE CAN ANSWER <u>QUERIES</u>

A QUERY CONSISTS OF A POINT $x \in \mathbb{R}^n$

THE RESPONSE TO A QUERY $x$

IS THE FAMILY OF NONZERO COEFFICIENTS

$\lambda(x, y)$.

THE WORK TO ANSWER A QUERY IS $\leq C \log N$

Thm 5: Let $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

Then there exist subsets $S_1, S_2, \dots, S_K \subset E$ such that:

$$\boxed{\begin{array}{l} K \leq CN \quad \& \\ \#(S_k) \leq C \quad \text{for } k = 1, \dots, K \end{array}}$$

$$\boxed{\begin{array}{l} \text{For any } f : E \to \mathbb{R}, \text{ we have} \\ \|f\|_{X(E)} \sim \max\limits_{k=1,\dots,K} \|(f|_{S_k})\|_{X(S_k)} \end{array}}$$

"SHARP FINITENESS THM"

MOREOVER, THE SETS $S_1, \dots, S_K$

CAN BE COMPUTED FROM $E$

USING AT MOST $CN \log N$ ops

& AT MOST $CN$ STORAGE.

---

RECALL, IF $\#(S) \leq C$,

THEN WE CAN COMPUTE

THE ORDER OF MAGNITUDE

OF $\| f \|_{X(S)}$

BY LINEAR ALGEBRA,

THANKS TO THE WHITNEY EXT. THM.

# REMOVING

# OUTLIERS

GIVEN

$f : E \to \mathbb{R}$ with $\#(E) = N < \infty$,

and given an integer $Z \leq N$,

WE WANT TO FIND $S \subset E$

WITH $\#(S) \leq Z$

AND

$\| f \|_{X(E \setminus S)}$ AS SMALL AS POSSIBLE.

# EASIER PROBLEM:

COMPUTE $S_Z \subset E$ SUCH THAT

$$\boxed{\#(S_Z) \leq CZ}$$

and such that

For any $S \subset E$ with $\#(S) \leq Z$,

we have

$$\|f\|_{X(E \setminus S_Z)} \leq C \|f\|_{X(E \setminus S)}.$$

WE CALL SUCH $S_Z$ "ALMOST OPTIMAL"

THM 6: GIVEN $f: E \to \mathbb{R}$ with $\#(E) = N < \infty$.

Using at most $CN^2 \log N$ ops & at most $CN$ storage, we can compute an enumeration

$x_1, x_2, \ldots, x_N$

of the elements of $E$, such that for any $Z$ $(Z \leq N/C)$, the set $S_Z = \{x_1, \ldots, x_{CZ}\}$ is almost optimal.

"REMOVING OUTLIERS".

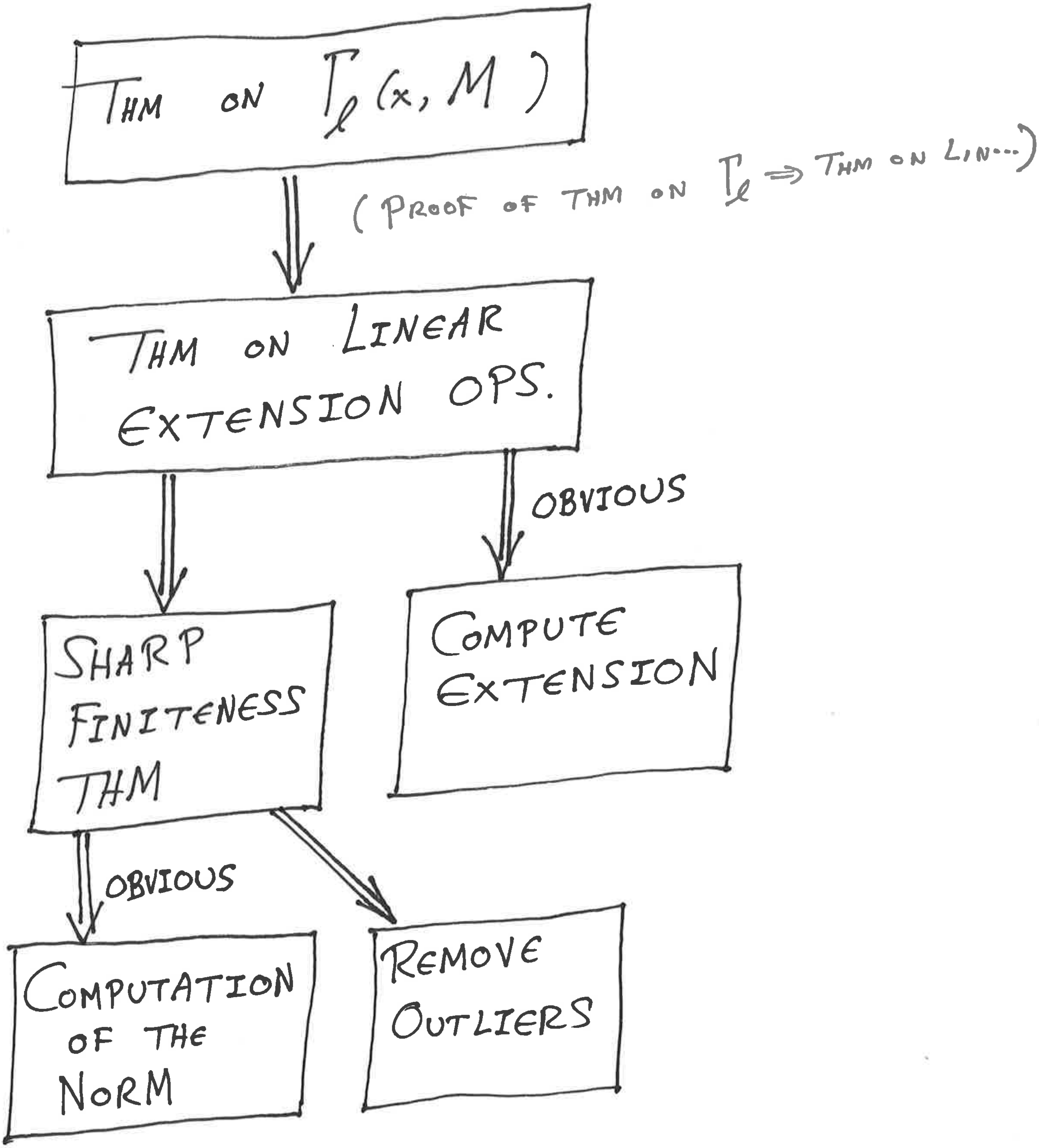
How Thms 1 … 6

are Logically

Related

To

One Another


THM ON $\Gamma_\ell(x, M)$
(PROOF OF THM ON $\Gamma_\ell$ ⇒ THM ON LIN…)
THM ON LINEAR EXTENSION OPS.
OBVIOUS
SHARP FINITENESS THM
COMPUTE EXTENSION
OBVIOUS
COMPUTATION OF THE NORM
REMOVE OUTLIERS

SHARP FINITENESS THM

$\Downarrow$

REMOVE OUTLIERS

RECALL,

- $\|f\|_{X(E)} \sim \max_{k=1,\cdots,K} \|f\|_{X(S_k)}$
- $S_1, \cdots, S_K \subset E$
- $K \leq CN$
- $\#(S_k) \leq C$, each $k$
- Can compute $S_1, \cdots, S_K$
- For each $k$, can compute $Y_k \sim \|f\|_{X(S_k)}$

# PSEUDOCODE TO ENUMERATE $E$

START WITH AN EMPTY LIST $\Lambda$.

1. COMPUTE $S_1, \ldots, S_K$, $Y_1, \ldots, Y_K$ as just explained.
2. Pick $\hat{k}$ to maximize $Y_{\hat{k}}$.
3. Append the elements of $S_{\hat{k}}$ to $\Lambda$.
4. Delete the elements of $S_{\hat{k}}$ from $E$.
5. If $E \neq \emptyset$, go to 1; if $E = \emptyset$, STOP

This algorithm terminates, and the resulting list $\Lambda$ is the enumeration of the elements of $E$ promised in Thm 6.

Let's see WHY ...

The algorithm terminates,
because $\#(E)$ shrinks
each time we execute steps ①…⑤.

---

$\Lambda$ is an enumeration of all the
elements of $E$, because as we delete
elements from $E$ we append them
to $\Lambda$. (We keep going until $E = \phi$.)

---

LET $S_Z$ be the contents of the list $\Lambda$ just after we have executed steps 1 ... 5 for the $Z^{\text{th}}$ time.

We must show that

$$\boxed{\#(S_Z) \le CZ}$$

and

$$\boxed{\begin{array}{l}\text{If } S \subset E \text{ with } \#(S) \le Z, \text{ then} \\ \|f\|_{X(E \setminus S_Z)} \le C \|f\|_{X(E \setminus S)}.\end{array}}$$

($S_Z$ is "ALMOST OPTIMAL")

It's obvious that

$$\#(S_Z) \le CZ,$$

because every time we execute steps ① … ⑤, we append at most $C$ elements to $\Lambda$.

THE NON-TRIVIAL ASSERTION IS THAT

(*) $\boxed{\begin{array}{l} \|f\|_{X(E \setminus S_Z)} \leq C \|f\|_{X(E \setminus S)} \\ \text{WHENEVER} \\ \#(S) \leq Z. \end{array}}$

We will prove this by induction on $Z$. For $Z=0$, we have $S_Z = S = \phi$, and our assertion holds trivially.

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ASSUMING (*) FOR $Z-1$, WE WILL PROVE IT FOR $Z$

WHY $\|f\|_{X(E\setminus S_2)} \le C\|f\|_{X(E\setminus S)}$

Two cases:

$$\boxed{\begin{array}{l} \text{① } S \text{ IS A "WEAK COMPETITOR",} \\ \text{i.e., } \|f\|_{X(E\setminus S)} \sim \|f\|_{X(E)} \end{array}}$$

$$\boxed{\begin{array}{l} \text{② } S \text{ IS A "STRONG COMPETITOR",} \\ \text{i.e.,} \\ \|f\|_{X(E\setminus S)} \ll \|f\|_{X(E)} \end{array}}$$

IF $S$ IS A WEAK COMPETITOR,

THEN OBVIOUSLY

$$\|f\|_{X(E \setminus S_Z)} \leq \|f\|_{X(E)} \sim \|f\|_{X(E \setminus S)}$$

SO

$$\|f\|_{X(E \setminus S_Z)} \leq C \|f\|_{X(E \setminus S)},$$

SO INDUCTION GOES THROUGH

IN THIS CASE.

## SUPPOSE $S$ IS A STRONG COMPETITOR

$$\boxed{\begin{array}{l}\text{MUST PROVE} \\ \|f\|_{X(E \setminus S_Z)} \leq C\|f\|_{X(E \setminus S)}.\end{array}}$$

Let $S_1, \dots, S_K$, $\varphi_1, \dots, \varphi_k$, $\hat{k}$ be as in the first time we execute steps ① ... ⑤ of our algorithm.

Then $\boxed{S_{\hat{k}} \subset S_Z}$.

We will see that $\boxed{S \cap S_{\hat{k}} \neq \emptyset.}$

WHY $S \cap S_{\hat{k}} \neq \emptyset$.

Suppose $S \cap S_{\hat{k}} = \emptyset$.

Then $S_{\hat{k}} \subset E \setminus S$, hence

$$\boxed{\begin{aligned} Y_{\hat{k}} &\sim \|f\|_{X(S_{\hat{k}})} \leq \|f\|_{X(E \setminus S)} \\ &\ll \|f\|_{X(E)} \sim \max_{k=1,\dots,K} \|f\|_{X(S_k)} \\ &\sim \max_{k=1,\dots,K} Y_k = Y_{\hat{k}} \end{aligned}}$$

So $Y_{\hat{k}} \ll Y_{\hat{k}}$, <u>CONTRADICTION.</u>

# Completing the Induction on $Z$

Want to show that

$$\boxed{\|f\|_{X(E \setminus S_Z)} \leq C \|f\|_{X(E \setminus S)}.}$$

Let $\tilde{E} = E \setminus S_{\hat{k}}$

$\tilde{f} = f|_{\tilde{E}}$

$\tilde{S} = S \cap \tilde{E}.$

Then $\boxed{\#(\tilde{S}) \leq Z - 1}$

because $S \cap S_{\hat{k}} \neq \emptyset.$

LET $\tilde{S}_{Z-1}$ BE THE CONTENTS OF THE LIST $\tilde{\Lambda}$, OBTAINED BY EXECUTING STEPS ①···⑤ $(Z-1)$ TIMES, STARTING FROM THE INPUT $\tilde{E}, \tilde{f}$ INSTEAD OF $E, f$.

THEN ONE CHECKS THAT

$$S_Z = S_{\hat{k}} \cup \tilde{S}_{Z-1},$$

SO THAT

$$E \setminus S_Z = \tilde{E} \setminus \tilde{S}_{Z-1}$$

AND

$$E \setminus S = \tilde{E} \setminus \tilde{S}.$$

INDUCTION HYPOTHESIS
( (*) FOR $Z-1$ ) $\Longrightarrow$

$$\|\tilde{f}\|_{X(\tilde{E} \setminus \tilde{S}_{Z-1})} \leq C \|\tilde{f}\|_{X(\tilde{E} \setminus \tilde{S})}.$$

THEREFORE,

$$\boxed{\|f\|_{X(E\setminus S_Z)} \leq C\|f\|_{X(E\setminus S)}}$$

FOR THE SAME CONST. $C$

AS IN THE INDUCTION HYP.

THIS COMPLETES OUR INDUCTION

ON $Z$, PROVING THAT

THE ALGORITHM WORKS.

WE'VE FINISHED

WITH

OUTLIERS.

ON TO THE NEXT SUBJECT...

LINEAR EXTENSION OPERATOR OF BOUNDED DEPTH

$\Downarrow$

SHARP FINITENESS THM.

# THE PLAN

- USE WELL-SEPARATED PAIRS DECOMPOSITION

- BRING IN LINEAR EXTENSION OPERATORS OF BOUNDED DEPTH

- PUT IT ALL TOGETHER

# WARM - UP EXERCISE

RECALL THE WELL-SEPARATED PAIRS DECOMPOSITION:

Given $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

Then $E \times E \setminus$ DIAGONAL

is partitioned into

Cartesian products

$$E'_\nu \times E''_\nu \quad (\nu = 1, \dots, \nu_{max})$$

where ...

$$\boxed{\nu_{max} \leq CN}$$

and

$$\boxed{\begin{array}{l} \mathrm{DIST}\,(E_\nu', E_\nu'') \geq 10^3 \left[ \mathrm{DIAM}(E_\nu') + \mathrm{DIAM}(E_\nu'') \right] \\ \text{for each } \nu = 1, \dots, \nu_{max} \end{array}}$$

FOR EACH $\nu = 1, \dots, \nu_{MAX}$, WE

PICK A "REPRESENTATIVE"

$(x_\nu', x_\nu'') \in E_\nu' \times E_\nu''$.

RECALL,

CALLAHAN - KOSARAJU

COMPUTE SUCH

REPRESENTATIVES

USING AT MOST

$C N \log N$

COMPUTER OPS,

AND AT MOST

$CN$

MEMORY.

# IOU FROM LECTURE 2

Given $f: E \to \mathbb{R}$,

we compute the Lipschitz const.

$$\text{MAX} \left\{ \frac{|f(x) - f(y)|}{|x-y|} : x, y \in E,\ x \neq y \right\}$$

to within a 1% ERROR

in $\leq C N \log N$ computer ops

using the WSPD.

WE WILL PROVE THAT

$$\boxed{\begin{aligned} &\mathrm{MAX}\left\{ \frac{|f(x)-f(y)|}{|x-y|} : x,y\in E,\ x\neq y \right\} \\ &\leq (1.01)\cdot \mathrm{MAX}\left\{ \frac{|f(x_\nu')-f(x_\nu'')|}{|x_\nu'-x_\nu''|} : 1\leq \nu \leq \nu_{MAX} \right\}. \end{aligned}}$$

RHS CAN BE COMPUTED IN $\sim N$ COMPUTER OPS, ONCE WE KNOW THE REPRESENTATIVES $x_\nu'$, $x_\nu''$.

# Proof of the Inequality

Suppose $|f(x'_\nu) - f(x''_\nu)| \leq |x'_\nu - x''_\nu|$ for each $\nu$.

Must show that

$$|f(x') - f(x'')| \leq (1.01)\,|x' - x''|$$

for any $x', x'' \in E$ $\quad (x' \neq x'')$.

<u>Suppose not</u>. Pick a counterexample $(x', x'')$ with $|x' - x''|$ as small as possible.

So

$$|f(x') - f(x'')| > (1.01) \cdot |x' - x''|$$

but

$$|f(z) - f(w)| \leq (1.01)\,|z - w|$$

whenever $|z - w| < |x' - x''|$.

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Now $(x', x'') \in E \times E \setminus \text{DIAGONAL}$.

Pick $\nu$ S.T. $(x', x'') \in E'_\nu \times E''_\nu$.

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LET $(x'_\nu, x''_\nu) \in E'_\nu \times E''_\nu$

BE THE REPRESENTATIVE.

BOTH $x'$ and $x_\nu'$ belong to $E_\nu'$.

THEREFORE,

$$|x' - x_\nu'| \leq \mathrm{DIAM}(E_\nu').$$

Similarly,

$$|x'' - x_\nu''| \leq \mathrm{DIAM}(E_\nu'').$$

RECALL THAT

$$\mathrm{DIST}(E_\nu', E_\nu'') \geq 10^3 \cdot [\mathrm{DIAM}(E_\nu') + \mathrm{DIAM}(E_\nu'')].$$

CONSEQUENTLY,

$$|x' - x''| \geq 10^3 \cdot [\, |x' - x_\nu'| + |x'' - x_\nu''| \,].$$

In particular,

$$|x' - x_\nu'|, \ |x'' - x_\nu''| < |x' - x''|,$$

So

$$|f(x') - f(x_\nu')| \le (1.01) \cdot |x' - x_\nu'|$$

and

$$|f(x'') - f(x_\nu'')| \le (1.01)\, |x' - x_\nu''| .$$

Recall that we are assuming that

$$|f(x_\nu') - f(x_\nu'')| \le |x_\nu' - x_\nu''|$$

$$\le |x' - x''| + |x' - x_\nu'| + |x'' - x_\nu''|.$$

COMBINING THESE ESTIMATES, WE SEE THAT

$$|f(x') - f(x'')| \leq$$

$$|f(x') - f(x_\nu')| + |f(x_\nu') - f(x_\nu'')| + |f(x_\nu'') - f(x'')|$$

$$\leq$$

$$[(1.01)\,|x' - x_\nu'|] +$$

$$[\,|x' - x''| + |x' - x_\nu'| + |x'' - x_\nu''|\,]$$

$$+ [(1.01)\,|x'' - x_\nu''|]$$

$$\leq |x' - x''| + 3\,|x' - x_\nu'| + 3\,|x'' - x_\nu''|.$$

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RECALL THAT $|x' - x_\nu'| + |x'' - x_\nu''| \leq 10^{-3}\,|x' - x''|$.

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THEREFORE,

$$|f(x') - f(x'')| \leq (1 + 3 \times 10^{-3})|x' - x''|.$$

HOWEVER, WE SAW EARLIER THAT

$$|f(x') - f(x'')| > (1.01)\,|x' - x''|.$$

THIS CONTRADICTION PROVES OUR INEQUALITY.

WE HAVE SUCCEEDED IN COMPUTING LIPSCHITZ CONST'S.

WE'VE COMPLETED

THE

WARM-UP

EXERCISE !

RETURN TO THE PROOF

THAT

$\boxed{\text{LINEAR EXTENSION OPERATORS OF BOUNDED DEPTH}}$

$\Downarrow$

$\boxed{\text{SHARP FINITENESS THM.}}$

IN THE SAME SPIRIT

AS THE

WARM-UP EXERCISE

WE CAN PROVE

THE FOLLOWING

LEMMA ON

WHITNEY FIELDS.

# THE GLORIFIED WARM-UP EXERCISE

LEMMA:

- Let $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

- Let $E_\nu' \times E_\nu''$ $(\nu = 1, \dots, \nu_{MAX})$ be a well-separated pairs decomp. of $E \times E \setminus$ Diagonal.

- Let $(x_\nu', x_\nu'') \in E_\nu' \times E_\nu''$ $(\nu = 1, \dots, \nu_{MAX})$ be representatives.

- Let $M > 0$ be a real number

- Let $(P^x)_{x \in E}$ be a Whitney field

SUPPOSE THAT

$$\boxed{\begin{gathered}\left|\partial^\alpha\left(P^{x_\nu'}-P^{x_\nu''}\right)(x_\nu')\right| \le M\,|x_\nu'-x_\nu''|^{m-|\alpha|}\\ \text{for } |\alpha|\le m-1,\quad \nu=1,\dots,\nu_{max}\end{gathered}}$$

THEN

$$\boxed{\begin{gathered}\left|\partial^\alpha\left(P^{x}-P^{y}\right)(x)\right| \le CM\,|x-y|^{m-|\alpha|}\\ \text{for } |\alpha|\le m-1,\quad x,y\in E.\end{gathered}}$$

# BRINGING IN EXTENSION OPERATORS OF BOUNDED DEPTH

SUPPOSE $T: X(E) \to X$

IS A LINEAR EXTENSION

<u>OPERATOR OF BDD. DEPTH.</u>

GIVEN $x \in \mathbb{R}^n$, THE JET

$J_x(Tf)$ IS COMPUTED FROM

$f|_{S(x)}$, WHERE

$S(x) \subset E$ HAS AT MOST $C$ POINTS.

WE KNOW THAT

$$Tf = f \text{ on } E$$

AND THAT

$$\|Tf\|_X \leq C \|f\|_{X(E)}.$$

**LEMMA:** Let $f : E \to \mathbb{R}$,

and let $x, y \in E$.

Suppose $\| f \|_{S(x) \cup S(y)} \leq 1$.

Then

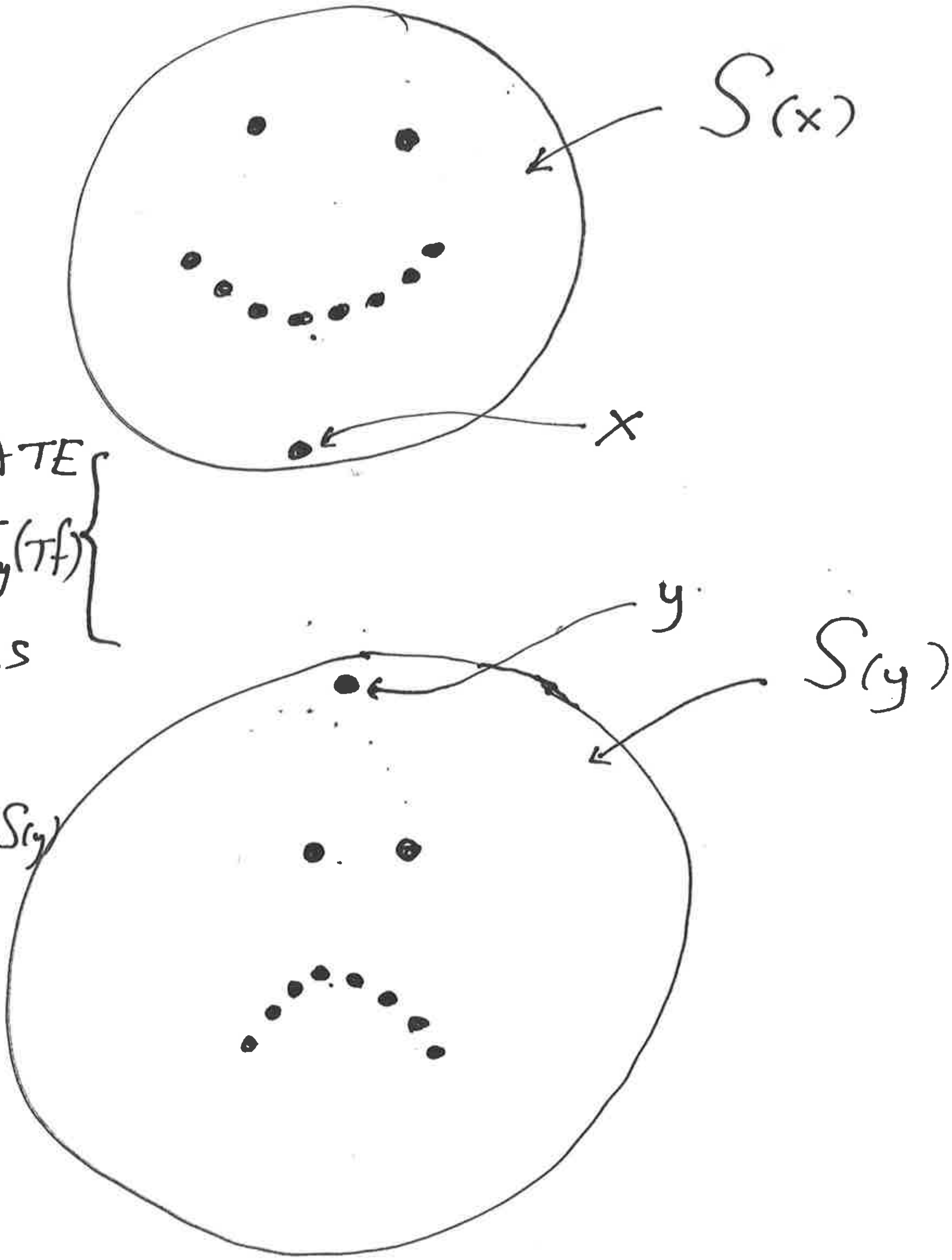
$$\boxed{\begin{gathered} \left| \partial^\alpha \left[ J_x(Tf) - J_y(Tf) \right](x) \right| \leq \\ C |x-y|^{m-|\alpha|} \\ \text{for } |\alpha| \leq m-1. \end{gathered}}$$

ESTIMATE

$J_x(Tf) - J_y(Tf)$

IN TERMS OF

$\|f\|_{S(x)\cup S(y)}$



## PROOF OF THE LEMMA:

SINCE $\|f\|_{S(x) \cup S(y)} \leq 1$,

there exists $F \in X$ such that

$$\boxed{\|F\|_X \leq 2 \text{ and } F = f \text{ on } S(x) \cup S(y).}$$

LET $\tilde{f} = F\big|_E$.

Then

$$\boxed{\begin{gathered}\tilde{f} = f \text{ on } S(x) \cup S(y) \\ \text{AND} \\ \|\tilde{f}\|_{X(E)} \leq 2.\end{gathered}}$$

LET $\tilde{F} = T\tilde{f}$.

BECAUSE $\tilde{f} = f$ on $S(x)$,

WE KNOW THAT

$$\boxed{J_x(\tilde{F}) = J_x(Tf).}$$

SIMILARLY,

$$\boxed{J_y(\tilde{F}) = J_y(Tf).}$$

BECAUSE $T : X(E) \to X$ IS BDD,

WE HAVE ALSO

$$\boxed{\|\tilde{F}\|_X \le C \|\tilde{f}\|_{X(E)} \le C'.}$$

CONSQUENTLY, for $|\alpha| \leq m-1$, WE HAVE

$$|\partial^\alpha(J_x(Tf) - J_y(Tf))(x)|$$

$$= |\partial^\alpha(J_x(\tilde{F}) - J_y(\tilde{F}))(x)|$$

$$\leq C'|x-y|^{m-|\alpha|}$$

by Taylor's Thm.

The proof of the LEMMA is complete.

# PUTTING IT ALL TOGETHER

LET $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

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MAKE A WELL-SEPARATED PAIRS DECOMPOSITION WITH REPRESENTATIVES

$(x_\nu', x_\nu'') \in E_\nu' \times E_\nu''$,

$\nu = 1, \dots, \nu_{max}$.

---

RECALL, $\nu_{MAX} \leq CN$.

LET $T: X(E) \to X$

BE A LINEAR EXTENSION OP.

OF BDD. DEPTH.

So $Tf = f$ on $E$.

$$\boxed{\begin{array}{l}\text{For each } x \in \mathbb{R}^n, \\ J_x(Tf) \text{ is determined by } f|_{S(x)} \\ \text{where } S(x) \subset E \text{ has} \leq C \text{ points.}\end{array}}$$

# GOAL: SHARP FINITENESS THM.

WILL DEFINE SUBSETS

$S_1, \dots, S_{\nu_{MAX}} \subset E$

WITH

$$\boxed{\nu_{MAX} \leq CN, \quad \#(S_\nu) \leq C \text{ for each } \nu.}$$

WE WILL THEN SHOW THAT

$$\boxed{\|f\|_{X(E)} \leq C \max_{\nu=1,\dots,\nu_{MAX}} \|f\|_{X(S_\nu)} \quad \text{for any } f \in X(E)}$$

FOR $\nu = 1, \cdots, \nu_{MAX}$, WE DEFINE

$$S_\nu = S(x'_\nu) \cup S(x''_\nu),$$

WHERE $(x'_\nu, x''_\nu)$ ARE

THE REPRESENTATIVES

FROM THE WSPD.

WE ALREADY KNOW THAT

$$\nu_{MAX} \leq CN$$

and that

$\#(S_\nu) \leq C$ for each $\nu$.

IT REMAINS TO SHOW THAT

$$\|f\|_{X(E)} \leq C \cdot \max_{\nu} \|f\|_{X(S_\nu)}$$

for any $f \in X(E)$.

We will suppose that

$$\|f\|_{X(S_\nu)} \leq 1 \quad \text{for each } \nu,$$

and prove that

$$\|f\|_{X(E)} \leq C.$$

For $x \in E$, DEFINE

$$P^x = J_x(Tf).$$

SINCE $Tf = f$ on $E$, WE HAVE

$$\boxed{P^x(x) = f(x) \quad \text{for all } x \in E.}$$

RECALL,

$$S_\nu = S(x_\nu') \cup S(x_\nu'') \text{ by DEF.},$$

and

$$\|f\|_{X(S_\nu)} \leq 1 \quad \text{by ASSUMPTION.}$$

APPLYING THE PRECEDING LEMMA,

WE FIND THAT

$$\boxed{\begin{gathered} \left| \partial^\alpha \left[ J_{x_\nu'}(Tf) - J_{x_\nu''}(Tf) \right](x_\nu') \right| \\ \leq C \left| x_\nu' - x_\nu'' \right|^{m-|\alpha|} \\ \text{for } |\alpha| \leq m-1. \end{gathered}}$$

That is,

$$\left|\partial^\alpha\left(P^{x_\nu'} - P^{x_\nu''}\right)(x_\nu')\right| \leq C\left|x_\nu' - x_\nu''\right|^{m-|\alpha|}$$

for $|\alpha| \leq m-1$, $\nu = 1, \ldots, \nu_{MAX}$.

The GLORIFIED WARM-UP EXERCISE now shows that

$$\left|\partial^\alpha\left(P^x - P^y\right)(x)\right| \leq C'|x-y|^{m-|\alpha|}$$

for $|\alpha| \leq m-1$, $\nu = 1, \ldots, \nu_{MAX}$

Also, because

$$P^x = J_x(Tf),$$

$P^x$ is DETERMINED BY $f|_{S(x)}$

and

$$\|f\|_{X(S_\nu)} \leq 1,$$

one shows easily that

$$\boxed{|\partial^\alpha P^x(x)| \leq C}$$

for $|\alpha| \leq m-1$, $x \in E$.

(WE OMIT THE DETAILS)

So

$$|\partial^\alpha P^x(x)| \leq C \quad \text{for } x \in E, \ |\alpha| \leq m-1$$

and

$$|\partial^\alpha (P^x - P^y)(x)| \leq C|x-y|^{m-|\alpha|}$$

for $x, y \in E$, $|\alpha| \leq m-1$.

WHITNEY'S EXTENSION THM $\Rightarrow$

$\exists\ F \in X$ with $\|F\|_X \leq C'$,

SUCH THAT

$J_x(F) = P^x$ for all $x \in E$.

In particular,

since $P^x(x) = f(x)$ for all $x \in E$,

we have

$$F = f \text{ on } E$$

with $\|F\|_X \leq C'$.

Therefore, by definition,

$$\|f\|_{X(E)} \leq C'.$$

So, ASSUMING THAT

$$\|f\|_{X(S_\nu)} \leq 1 \text{ for } \nu = 1, \cdots, \nu_{MAX},$$

WE HAVE PROVEN THAT

$$\|f\|_{X(E)} \leq C'.$$

THIS COMPLETES THE PROOF THAT LINEAR EXT. OP. OF BDD DEPTH

$\Downarrow$

SHARP FINITENESS THM

FINALLY, WE SAY A FEW WORDS ABOUT THE IMPLICATION

$$\boxed{\text{THM ON } T_\ell(x, M)}$$

$$\Downarrow$$

$$\boxed{\text{LINEAR EXTENSION OPS. OF BOUNDED DEPTH}}$$

RECALL THAT THE THM ON $\Gamma_\ell(x, M)$

ASSERTS THAT

FOR LARGE ENOUGH $\ell_*$, $C$

DETERMINED BY $m, n$,

WE HAVE

$$\Gamma_{\ell_*}(x, M) \subset \Gamma(x, CM),$$ WHERE

$$\Gamma(x, M) = \{ J_x(F) : \|F\|_X \leq M,\ F = f \text{ on } E \}.$$

EQUIVALENTLY, THAT THM

ASSERTS THAT

$\boxed{\text{GIVEN } P \in \Gamma_{\ell_*}(x, M) \text{ THERE EXISTS } F \in X \text{ S.T. } \|F\|_X \leq CM \text{ AND } F = f \text{ on } E.}$

IN PARTICULAR:

(!) $\boxed{\text{IF } \Gamma_{\ell_*}(x, M) \neq \emptyset, \text{ then there exists } F \in X \text{ s.t. } \|F\|_X \leq CM \text{ AND } F = f \text{ on } E.}$

OUR PROOF OF THESE RESULTS

IS

CONSTRUCTIVE.

IN PARTICULAR, BY FOLLOWING OUR ARGUMENT CAREFULLY (& DOING A LITTLE EXTRA WORK)

ONE CHECKS THAT THE MAP

$f \longmapsto$ The $F$ PRODUCED BY OUR PROOF OF (!)

IS A LINEAR EXTENSION OP.

OF BOUNDED DEPTH.

LET'S JUST BELIEVE IT.

THE MORAL OF THE STORY IS THAT ALL OUR RESULTS ON $C^m(\mathbb{R}^n)\big|_E$ ($E$ FINITE) WILL FOLLOW IF WE CAN SHOW THAT

$$\Gamma_{l_*}(x, M) \subset \Gamma(x, CM).$$

THE NEXT 2 LECTURES WILL SKETCH THE PROOF OF THAT.

THANK YOU!